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PERFORMANCE EVALUATION OF AN INVERSE INTEGRAL EQUATION METHOD APPLIED TO TURBOMACHINE CASCADES

D. MARTIN

Brown Boveri Ltd. Switzerland, Bettstenstr. 12, CH-8305 Dietlikon, Switzerland

AND

M. RIBAUT

Brown Boveri Ltd. Switzerland, Bahnhofstr. 38, CH-5430 Wettingen, Switzerland

SUMMARY

An improved formulation of the inverse integral equation method proposed in Reference 1 is presented which allows, in particular, a well-posed problem to be ensured. The corresponding computation code is tested in an exhaustive manner for axial and radial compressor and turbine cascades. The agreement between the velocity field obtained with the inverse method and that resulting from a direct calculation is examined for subsonic, transonic and supersonic flows. Accuracy and reliability of the solution to the boundary condition problem are excellent for the subsonic and transonic flows. However, for the supersonic flow, the application of the method seems to be limited by the use of elementary solutions of the Laplace operator.

INTRODUCTION

The development of aerodynamic inverse design procedures has its origin in the optimization process of transonic aircraft performance. Accordingly, the best results are obtained today in this area[2] and only a few methods are available for application to the rotating mixed-flow cascade of a turbomachine. This situation is especially regrettable as the possibility of developing a shock-free transonic flow is enhanced by the presence of casing walls, which allow a further improvement of the compatibility between the double throated streamtubes and the blade passage. On the other hand, the possibility of linearizing the flow equations considerably reduces for the cascade flow, which may be of little importance for methods using the concept of local approximation. However, by giving up the elementary solutions to the wave equation, one faces again the essential and still unresolved problem of introducing the causality in an integral method. For the subsonic flow, the velocity field is usually obtainted by superposing a field of sources (elementary solutions of the Laplace operator satisfying the Poisson equation at one point) on the singularities arising from the boundary conditions. If the strength of the sources is calculated from a preceding solution, the boundary singularities result from a system of linear equations exactly as for the incompressible fluid flow. This technique of iteration has proved also to be efficient for the shock-free transonic aerofoil flow[3] and should apply with at least equal success to the transonic cascade flow. Indeed, in the borderline case of zero pitch to chord ratio, the periodic part of the velocity field induced by a

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source or vortex row vanish and only a symmetric part remains downstream (after having added the unperturbed flow). This parabolic behaviour of the velocity field could even justify the application of the above-mentioned elliptic elementary solutions to the slightly supersonic cascade flow.

In the first part of this paper, an improved version of the integral equation method described in Reference 1 is presented and some new aspects and problems discussed. In the second part, an exhausitve evaluation of the possibilities and limitations of the computation code is presented for a number of axial and radial compressor and turbine cascades working in the subsonic, transonic and supersonic flow ranges.

BASIC EQUATIONS

Consider the compressible and irrotational flow through a cascade situated on an arbitrary surface of revolution measured by the meridional co-ordinate m and the angular co-ordinate φ. The kinematical condition formulated at the point z of the profile surface S may be written

$$(w_m \cos\alpha' + w_\varphi \sin\alpha')(z) = \lambda\bar{\gamma}(z), \tag{1}$$

with $\alpha' = \alpha + (0{\cdot}5 - \lambda)\pi$, α being the angle between the tangent to the profile surface and the peripheral direction. Depending on whether equation (1) refers to the normal or tangential direction, λ takes the value zero or one half. The divergence of the velocity field may be derived from the isentropic flow equations:

$$q = \frac{w^2}{a^2}\left(\frac{\mathrm{d}w}{\mathrm{d}s} - \frac{r\omega^2}{w}\frac{\mathrm{d}r}{\mathrm{d}s}\right) - w_m\frac{1}{h}\frac{\mathrm{d}h}{\mathrm{d}m} \tag{2}$$

and accounts for the fluid compressibility and the variation of the channel height h. As q depends in a non-linear way on the velocity field, it is advantageous to calculate it from a preceding solution, thus preserving the linear character of the relationship between the relative velocity components w_m, w_φ and the vortex strength $\bar{\gamma}(x)$ on the profile surface:

$$\begin{aligned}(w_m, w_\varphi)(z) = {} & \frac{r_1}{r_z} w_1(\sin\alpha_1, \cos\alpha_1) - (0, r_z\omega) \\ & + \frac{N}{2\pi r_z}\left\{\int_S (K_1, K_2)[z, x(s)]\bar{\gamma}[x(s)]\,\mathrm{d}s\right. \\ & \left. + \iint_F (K_2, -K_1)[z, \tilde{x}(m,\varphi)]q[\tilde{x}(m,\varphi)]\,\mathrm{d}mr\mathrm{d}\varphi\right\},\end{aligned} \tag{3}$$

w_1, α_1 representing the unperturbed flow given upstream of the surface of revolution considered and ω being the angular velocity of the cascade. The geometrical kernels K_1, K_2 are given in Reference[3]. Interpolating $\bar{\gamma}$ over L points on S and the divergence q over M points of the domain F wetted by the fluid, equation (1) can be formulated for L-1 mid-points, yielding, together with the Kutta–Joukowsky condition, the linear equation system of the direct prolem:

$$\mathbf{D}(S)\mathbf{d} \quad - [w_1\mathbf{b}(S) + \omega\mathbf{c}(S) + \mathbf{E}(S)\mathbf{e}], \tag{4}$$

where $\mathbf{d}, \mathbf{e}$ are the vectors interpolating $\bar{\gamma}$ and q, respectively; $\mathbf{D}$ and $\mathbf{E}$ are matrices depending only on the profile geometry and $\mathbf{b}, \mathbf{c}$ are the following vectors:

$$\mathbf{b} = [\sin(\alpha'(x_1) + \alpha_1), \ldots\ldots, \sin(\alpha'(x_{L-1}) + \alpha_1), 0],$$

$$\mathbf{c} = [(r\sin\alpha')(x_1), \ldots\ldots, (r\sin\alpha')(x_{L-1}), 0].$$

The convergence of this iterative solution method has been shown to be closely related to the absence of shocks in the flow field.[3] However, as the Mach number increases, the compatibility between the local and the boundary conditions reduces, and the minimal convergence interval attained in the course of the iterations increases. The proper way to improve this situation would be to solve the causality problem by calculating the velocity field. Thus, the possibility of having discontinuities, and therefore only single throated streamtubes, would ensure again the convergence of the calculation. But this is extremly difficult to realize if one is not intending to change the compressibility law.[4] A first step in this direction consists of calculating the velocity gradient in equation (2), defining the source, with an up-winded difference scheme. The introduction of this local causality into the direct and inverse computation codes substantially improves the convergence and the agreement of the solution with experiment.[3]

By solving the inverse problem, a vorticity distribution $\mathbf{d}^*$ is prescribed together with some initial values for S, w_1 and ω. In order to satisfy the kinematical condition, we introduce new values $S+\Delta S, w_1+\Delta w_1$ and $\omega+\Delta\omega$ in (4):

$$\mathbf{D}(S+\Delta S)\mathbf{d}^* = -(w_1+\Delta w_1)\mathbf{b}(S+\Delta S)-(\omega+\Delta\omega)\mathbf{c}(S+\Delta S)-\mathbf{E}(S+\Delta S)\mathbf{e}. \tag{6}$$

This procedure is justified by the fact that, for a given source distribution, each vortex distribution $\mathbf{d}$ depends linearly on w_1 and ω, as can be seen from (4). Applying a Taylor expansion to (6) and neglecting second and higher order derivatives, one obtains the equation system of the inverse problem:

$$[\mathbf{D}'(S)\mathbf{d}^* + w_1\mathbf{b}'(S)+\omega\mathbf{c}'(S)+\mathbf{E}'(S)\mathbf{e}]\Delta S+\mathbf{b}(S)\Delta w_1+\mathbf{c}(S)\Delta\omega = \mathbf{D}(S)(\mathbf{d}^*-\mathbf{d}), \tag{7}$$

where $\mathbf{d}$ is the solution to (4) for the values of S, w_1, ω and $\mathbf{e}$ considered. Here, as for the direct problem, the source field e is calculated iteratively from a preceding solution. The resolution of the system of linear equations (7) yields the increments $\Delta S, \Delta w_1, \Delta\omega$ and, at convergence of the iterative process, the geometric profile one is looking for.

The best way to produce the variation ΔS of the geometry is certainly to shift the profile points in a direction normal to the profile surface. However, for practical application in turbomachinery, the choice of the direction φ seems to be more advantageous.[1] With this option, the component $\mathbf{D}'(S)\mathbf{d}^*\Delta S$ in a point z of the profile surface may be written:

$$r_z \sin\alpha'_z\left(\frac{\mathrm{d}\Delta\varphi}{\mathrm{d}s}\right)_z\int_S(\sin\alpha'_z K_1-\cos\alpha'_z K_2)\bar{\gamma}^*\mathrm{d}s+\int_S(\cos\alpha'_z K_1$$
$$+\sin\alpha'_z K_2)\bar{\gamma}^* r\cos\alpha'\frac{\mathrm{d}\Delta\varphi}{\mathrm{d}s}\mathrm{d}s+\int_S\left(\cos\alpha'_z\frac{\partial K_1}{\partial\varphi}+\sin\alpha'_z\frac{\partial K_2}{\partial\varphi}\right)\bar{\gamma}^*(\Delta\varphi-\Delta\varphi_z)\,\mathrm{d}s, \tag{8}$$

the function $\Delta\varphi$ being interpolated from L unknown values distributed on S. A similar expression could be obtained for $\mathbf{E}'\mathbf{e}\Delta S$. Neglecting the second term of (8), which accounts for the stretch of the abscissa s, and assuming Δw_1 and $\Delta\omega$ equal to zero in (6) leads to the equations used in reference 1. With the present formulation of the method the inverse problem can be well posed.[5] The presence of a stretch term in the Taylor development reduces the influence of the initial profile geometry and improves the convergence of the calculation procedure. The various possibilities of application to the two- and three-dimensional problems having been described in Reference 1, we close here the theoretical discussion.

NUMERICAL ASPECTS

The choice of a constant direction for the displacement of the profile points implies the fact that at least one point will practically move on the profile surface itself. In such a 'frozen' region

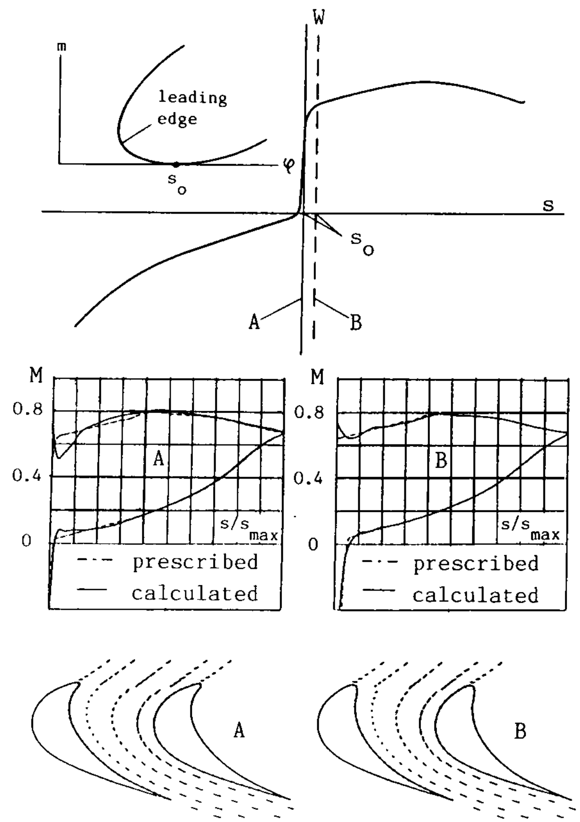

Figure 1. Feasibility of two velocity distributions A and B in the 'frozen' region around the point s_0

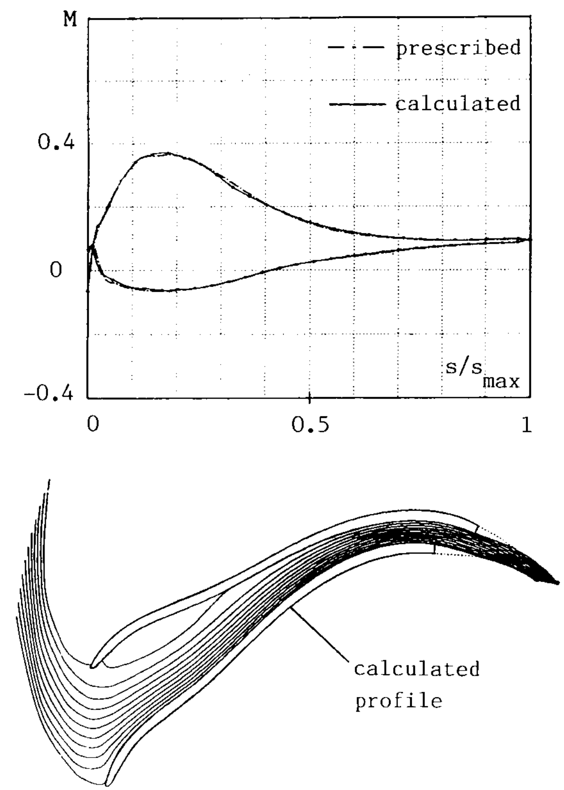

Figure 2. Calculated Mach number distribution and profile geometry of a radial turbine cascade with a reversal flow region

the velocity cannot be prescribed, but must be adjusted by shifting the origin s_0 of the abseissa s in the way illustrated in Figure 1. Another problem arises from the existence of low velocity stagnation regions. As can be seen from (7), the important term $\mathbf{D}'(S)\mathbf{d}^*$ may then considerably reduce or vanish. This situation is not catastrophic since the other terms, especially the one involving the unperturbed flow, are still present. However, for cascades presenting important regions of low velocity, the troubles may become significant. In such cases, a smoothing technique consisting of the piecewise interpolation of each $\Delta\varphi$ with parabolae has been shown to be efficient (see Figure 2). A last difficulty worth mentioning results from the degeneracy of the information in regions characterized by a small inclination of the velocity vector against the peripheral direction. Indeed, to prescribe the velocity at the pressure and suction sides is nearly equivalent to giving a mean channel velocity and its peripheral component. As a consequence, as the flow angle reduces, the more the determination of the meridional velocity component and therefore of the profile thickness becomes worse. The remedy is to reduce the error level of the computation code as much as possible or to prescribe the profile thickness at one point, at least.

The actual computation code works with about 50 to 100 basis intervals distributed over the profile surface, the radius of the leading edge defining the size of the first interval. For both the vortex distribution and the profile geometry a cubic Bessel interpolation[6] is used. The source (2) is calculated at each of about 25 meridional stations for seven Gauss points distributed, in the peripheral direction, over the channel width. Depending on the amount of change of the profile geometry, from one iteration to the next, the matrix $\mathbf{E}$ is calculated again or not, thus saving calculation time. In general 10 to 20 iterations are necessary to achieve a low error level (< 1 per

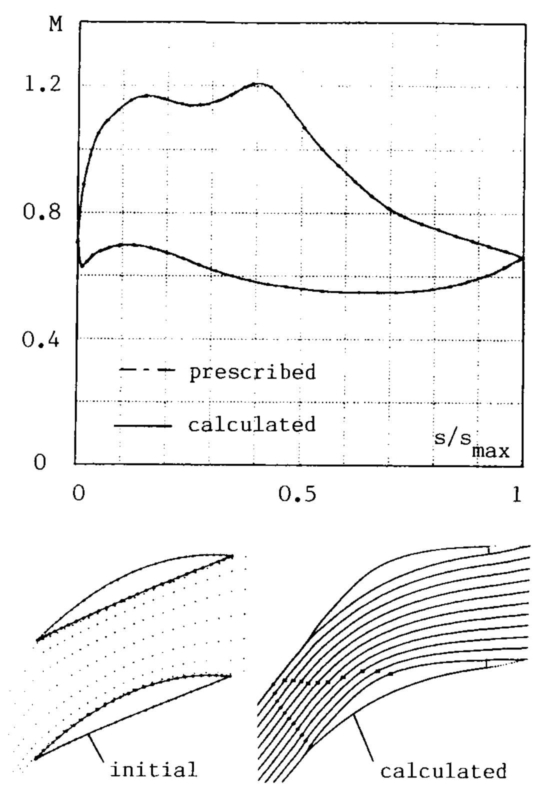

Figure 3. Mach number distribution and geometry of an axial compressor cascade

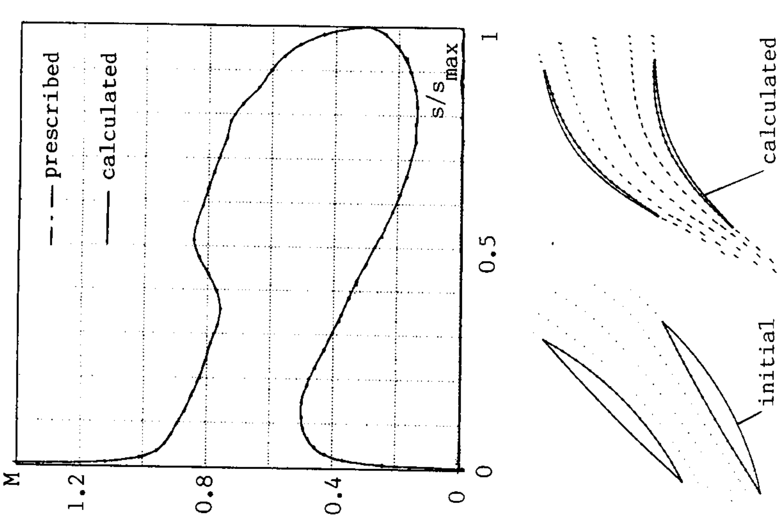

Figure 4. Mach number distribution and geometry of the shroud cascade of a radial compressor

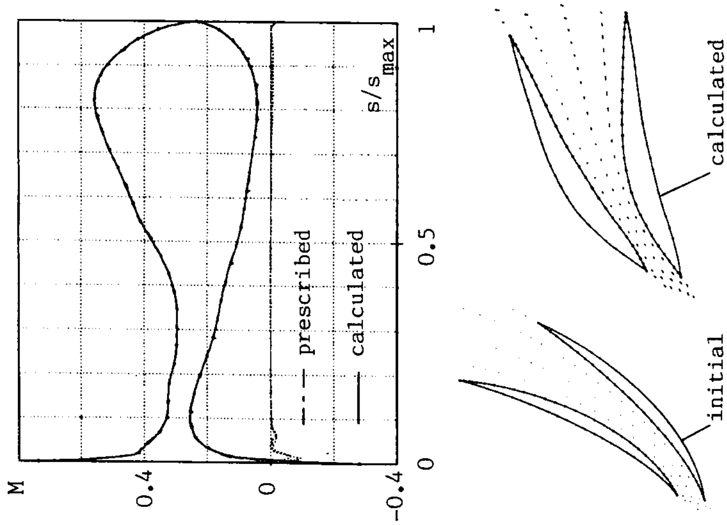

Figure 5. Mach number distribution and geometry of the hub cascade of a radial compressor

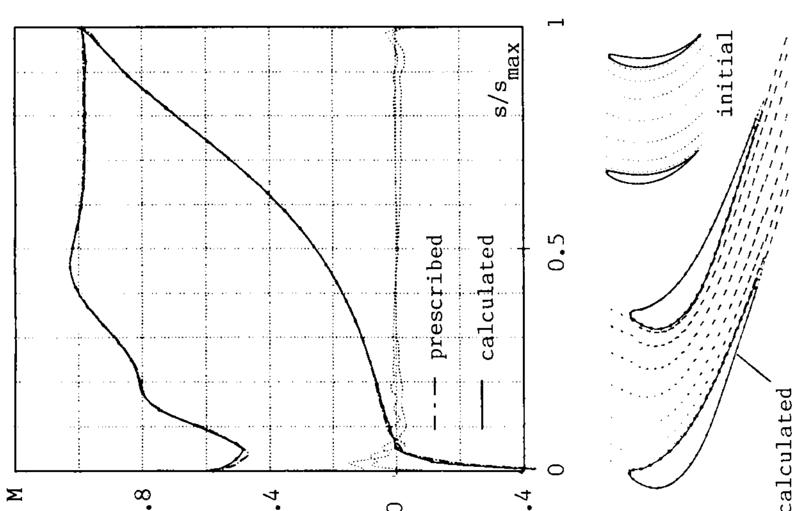

Figure 6. Mach number distribution and geometry of an axial turbine cascade

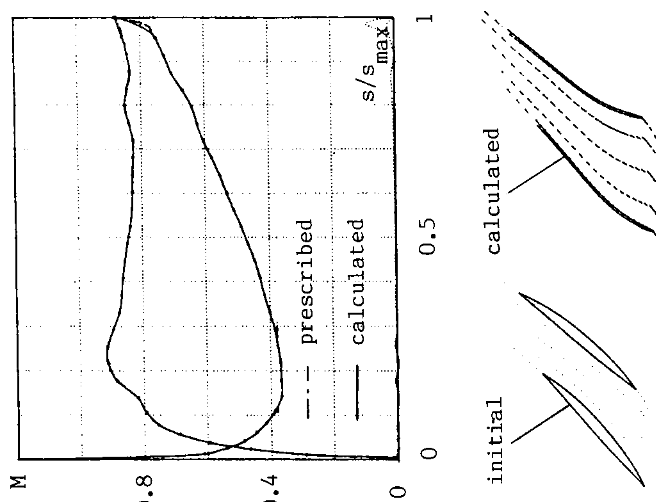

Figure 7. Mach number distribution and geometry of the shroud cascade of a radial turbine (the hub cascade is illustrated in Figure 2)

cent) and convergence interval (< 0·1 per cent) of the solution. The corresponding calculation time is situated between 200 and 300 seconds on an IBM-3081 computer. Further information about the numerical aspects of the inverse and direct solution methods may be found in Reference 1 and 3.

APPLICATION

The computational results presented in this section were obtained by prescribing a velocity distribution obtained from the direct calculation of a given cascade geometry. The initial profile geometry needed for the inverse calculation was generated by superposing a certain thickness distribution on a camber line defined by the inlet flow angle and an estimated outlet angle. In order to check the accuracy of the solution to the inverse boundary condition problem the velocity distribution at the surface of the final profile geometry was calculated with a direct method using the sources provided by the inverse calculation. For comparing this velocity distribution with the prescribed one, special attention was devoted to the compatibility between the direct and inverse computation codes, using the same basis functions and discretization of the line and surface integrals. Also, the value of the unperturbed flow was known; it has been shown to improve the convergence and accuracy of the solution to consider it as a variable. The results illustrated in Figures 2–7 demonstrate the capability of the present method to deal with any axial and radial compressor or turbine cascade of a turbomachine. The possibility of prescribing simultaneously the velocity distribution and the profile thickness by considering the height of the meridional channel as an unknown of the problem was described in Reference 1, and is illustrated in Figure 8.

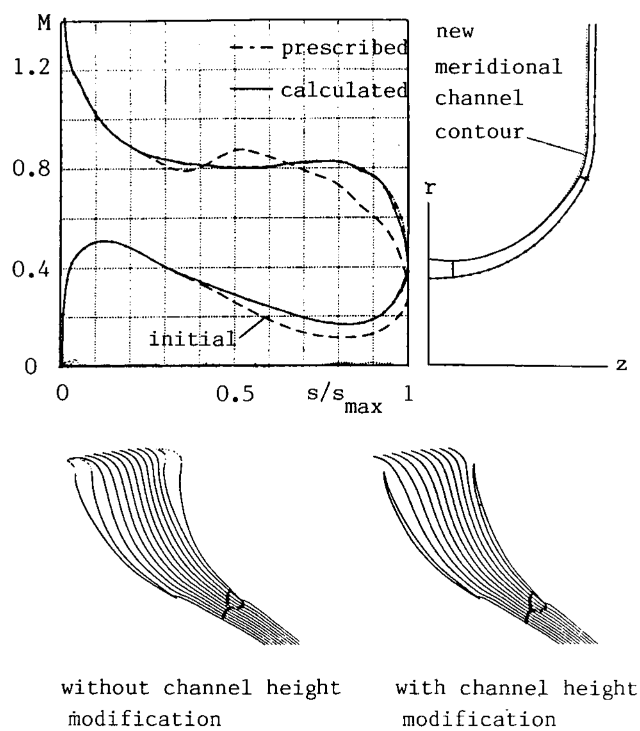

Figure 8. Mach number distribution and geometry of a radial compressor cascade calculated with prescribed blade thickness (see Figure 4) and variable channel height

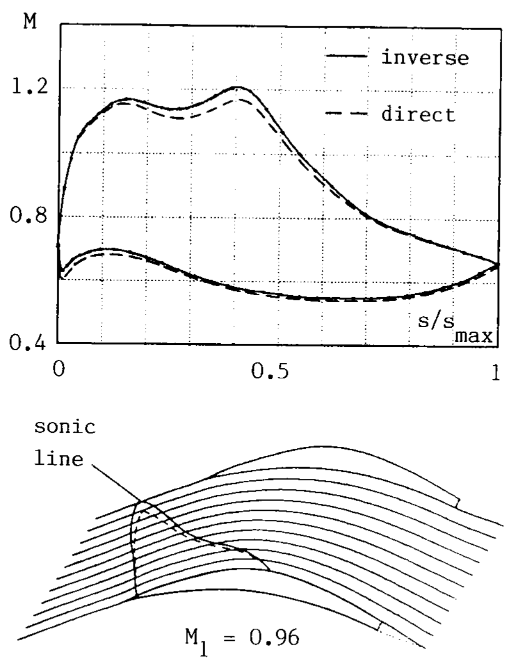

Figure 9. Transonic axial compressor flow obtained with the source fields of the inverse or direct methods

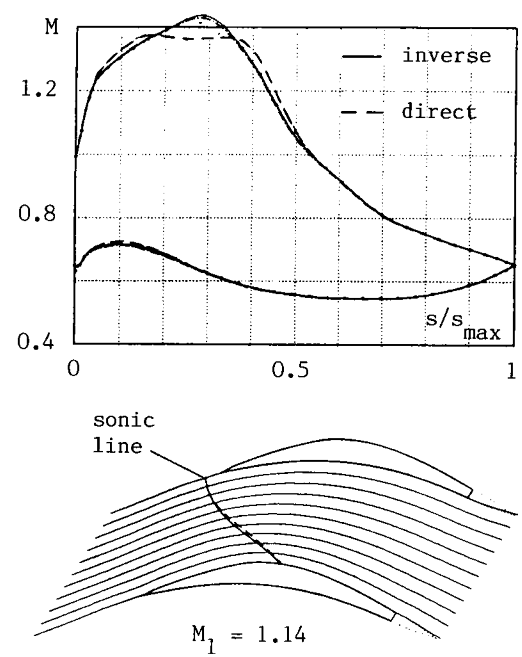

Figure 10. Supersonic axial compressor flow obtained with the source fields of the inverse or direct methods

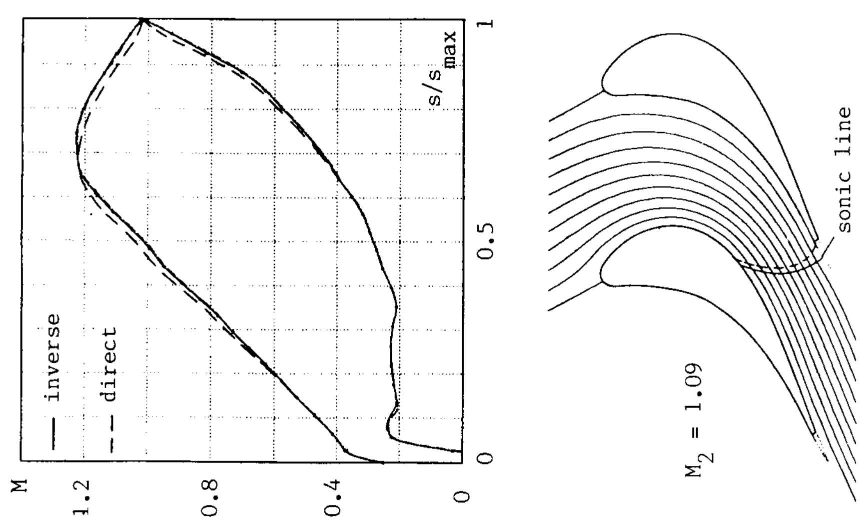

Figure 11. Supersonic axial turbine flow obtained with the source fields of the inverse or direct methods

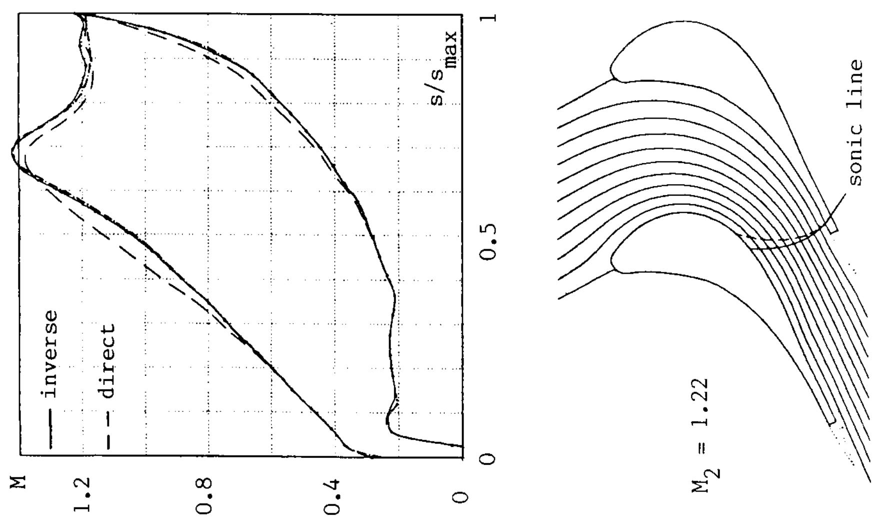

Figure 12. Supersonic axial turbine flow obtained with the source fields of the inverse or direct methods

It will be seen that for this rotating radial cascade an unacceptable thickening of the blades in the outlet region can be avoided by a three-dimensional formulation of the inverse problem.

It remains to be seen whether a calculated velocity distribution using sources provided by the direct problem agrees with the velocity prescribed for the inverse problem. Indeed, the non-linear part of the sources surrounding the profile is determined by the velocity prescribed at its surface, as can be seen from equation (2). The source field resulting from the solution to the direct problem, where the vortex strength at the profile surface strongly interacts with the sources in the course of the iterations, may therefore be different. It has been observed that for a subsonic flow there is practically no difference between the direct and inverse calculations. However, as the Mach number increases, the agreement between the two solutions of the strongly non-linear differential equation (2) reduces. This state of things is illustrated in Figures 9 to 12, for the transonic and supersonic flow through an axial compressor and an axial turbine. All calculations were stopped after a minimal mean convergence interval, obtained with the variation of the profile velocity from one iteration to the next, was attained. For all inverse and direct subsonic flow calculations and the transonic and supersonic compressor flows this interval was less than one per cent, whereas the two direct supersonic turbine flow calculations were stopped by 3 and 9 per cent, respectively. This bad convergence may result from the fact that the parabolic character induced in the flow field by the periodic disposition of the sources and vortices only extends in the axial direction, thus providing a better solution for the more meridional compressor flow.

CONCLUSIONS

The results presented in Reference 1 and in this paper demonstrate that coherent solutions to the direct and inverse problems can be obtained with the integral method for the subsonic and shock-free transonic flow through axial or radial turbomachine cascades. The agreement of the results of the direct calculation with hodograph solutions and experimental data[3] ensures the accuracy and reliability of the present inverse design method.

By making use of elementary solutions of the Laplace operator and avoiding any 'artificial viscosity' or 'fictitious gas' concept, the convergence of the solution is closely related to the compatibility between the local and boundary conditions of the real flow and the feasibility of the prescribed velocity. This allows, in particular, clear recognition of shock-free flow configurations.

Application of the method to supersonic compressor and turbine flows seems possible, but limited by the actual incapability to properly introduce domains of dependence and discontinuities in to the flow field.

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